Math 524 Exam 10 Solutions

1. Find all 2×2 complex matrices that are simultaneously diagonal, Hermitian, and unitary.

   Let $A = \left(\begin{smallmatrix} a & 0 \\ 0 & b \end{smallmatrix}\right)$, a diagonal matrix with complex entries. Its eigenvalues are precisely a, b. Because A is Hermitian, they must be real. Because A is unitary, they must each be of absolute value 1. There are exactly four matrices satisfying these conditions: $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. They satisfy the requirements, since for each of them $A^\dagger = A^T = A = A^{-1}$.

2. Find all 2×2 real matrices that are simultaneously symmetric and orthogonal.

   Let $A = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$, a symmetric matrix with real entries. Because A is symmetric and orthogonal (much as in problem 1), the eigenvalues must each be 1 or -1. Hence $|A| = \pm 1$. Consider first $|A| = 1$. Then $\frac{1}{1}\left(\begin{smallmatrix} c & -b \\ -b & a \end{smallmatrix}\right) = A^{-1} = A^\dagger = A = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$. Hence $a = c, b = -b$; the only such matrices are $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right), \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$. These two both satisfy the requirements, as in problem 1. Now consider $|A| = -1$. Then $\frac{1}{-1}\left(\begin{smallmatrix} c & -b \\ -b & a \end{smallmatrix}\right) = A^{-1} = A^\dagger = A = \left(\begin{smallmatrix} a & b \\ b & c \end{smallmatrix}\right)$. Hence $a = -c$. $-1 = |A| = -a^2 - b^2$, so we can write $A = \left(\begin{smallmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{smallmatrix}\right)$. This satisfies the requirements for every angle θ: it is obviously symmetric, and the columns (taken as vectors) are orthonormal hence the matrix is orthogonal.
   Note: this includes the diagonal matrices from problem 1, for $\theta = \pm \pi/2$.

3. Find all 2×2 complex matrices that are simultaneously anti-symmetric and unitary.

   Let $A = \left(\begin{smallmatrix} 0 & a \\ -a & 0 \end{smallmatrix}\right)$, an anti-symmetric matrix with complex entries. (note: $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = -\left(\begin{smallmatrix} a & c \\ b & d \end{smallmatrix}\right)$ yields $a = d = 0, b = -c$). Since A is unitary, its eigenvalues must each be of unit length, hence $|A| = a^2$ must be of unit length, so $|a| = 1$. We write $a = e^{i\theta}$; so $A = \left(\begin{smallmatrix} 0 & e^{i\theta} \\ -e^{i\theta} & 0 \end{smallmatrix}\right)$. For any θ, this satisfies the requirements: it is anti-symmetric by inspection, and unitary since the columns are orthonormal.

4. Find the maximum of $2x^2 + 4xy + 5y^2$ subject to $x^2 + y^2 = 1$.

   The quadratic function $f_A(x, y) = 2x^2 + 4xy + 5y^2$ is $\langle x|A|y\rangle$, for $A = \left(\begin{smallmatrix} 2 & 2 \\ 2 & 5 \end{smallmatrix}\right)$. This has eigenvalues $1, 6$. $\lambda = 1$ has eigenvector $\left(\begin{smallmatrix} 2 \\ -1 \end{smallmatrix}\right)$, which we divide by its length to satisfy the $x^2 + y^2 = 1$ condition: $\frac{1}{\sqrt{5}}\left(\begin{smallmatrix} 2 \\ -1 \end{smallmatrix}\right)$. $\lambda = 6$ has eigenvector $\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$, which we divide by its length to satisfy the $x^2+y^2 = 1$ condition: $\frac{1}{\sqrt{5}}\left(\begin{smallmatrix} 1 \\ 2 \end{smallmatrix}\right)$. By the methods show in the book, one of these must maximize our quadratic function. $f_A(2/\sqrt{5}, -1/\sqrt{5}) = 1, f_A(1/\sqrt{5}, 2/\sqrt{5}) = 6$. Hence the maximum is 6.